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An Investigation of Finite-Size Scaling for Systems with Long-Range Interaction: The Spherical Model

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A method is suggested for the derivation of finite-size corrections in the thermodynamic functions of systems with pair interaction potential decaying at large distances r as $r^{-d-\sigma}$, where d is the space dimensionality and $\sigma > 0$. It allows for a unified treatment of short-range ($\sigma = 2$) and long-range ($\sigma < 2$) interaction. The asymptotic analysis is illustrated by the mean spherical model of general geometry $L^{d-d'} \times \infty^{d'}$ subject to periodic boundary conditions. The Fisher–Privman equation of state is generalized to arbitrary real values of $d \geqslant \sigma$, $0 \leqslant d' < \sigma$. It is shown that the ε-expansion may be used to study the breakdown of standard finite-size scaling at the borderline dimensionalities.

KEY WORDS: Finite-size scaling; long-range interactions; spherical model; ε-expansion.

1. INTRODUCTION

Finite-size scaling for systems with long-range interactions decaying at large distances r as $r^{-d-\sigma}$, where d is the space dimensionality and $0 < \sigma \leqslant 2$ is a parameter, was first studied by Fisher and Privman.[(1)] Since the traditional approach[(2-6)] cannot be used in this case, Fisher and Privman[(1)] suggested an alternative approach based on direct evaluation of the closeness of certain d-fold sums to the corresponding d-dimensional integrals.

A crucial point of the approach[(2-6)] is the reduction of the

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d-dimensional problem to an effectively one-dimensional problem. In the case of short-range interactions this is achieved with the aid of the identity

$$(A+q^2)^{-1}=\int_0^\infty dt\,\exp(-At)\exp(-q^2t) \qquad (A\geqslant 0) \tag{1.1}$$

In this case of long-range interactions it is necessary to find an analogue of (1.1) for the propagator $[A+(q^2)^{\sigma/2}]^{-1}$ which appears in the evaluated finite sum instead of the left-hand side of (1.1). The way to the required generalization becomes clear if we regard the identity (1.1) as a Laplace transformation of the function $\exp(-At)$. Then we may write, instead of (1.1),

$$[A+(q^2)^{\sigma/2}]^{-1}=\int_0^\infty dt\,\mathscr{F}_{\sigma/2}(A,t)\exp(-q^2t) \tag{1.2}$$

With some ingenuity, a number of results can be obtained[7] even if the explicit form of $\mathscr{F}_{\sigma/2}(A, t)$ remains unknown, by just using the fact that it is a Laplace original of the left-hand side of (1.2). A further step in this approach[8] is the realization of the fact that $\mathscr{F}_{\sigma/2}(A, t)$ is expressible in terms of Mittag–Leffler-type functions, the analytical properties of which have been studied in detail.[9] By making use of these properties, we may easily study in the most general form finite-size effects in models with long-range interactions.

Traditionally, various ideas and methods in the theory of finite-size scaling are tested by the example of the mean spherical model,[10] or the lattice $\mathcal{O}(n)$ model in the $n\to\infty$ limit, which is the simplest model exactly solvable at any space dimensionality d. In refs. 7 and 8 we investigated finite-size scaling in the critical region of the d-dimensional mean spherical model in fully finite (block) geometry, with arbitrary interaction potential the Fourier transform of which has a leading-order long-wavelength exponent σ, $0<\sigma\leqslant 2$. In ref. 7 we reported a new derivation of the critical finite-size scaling equation for the spherical field in the Fisher–Privman form,[1] which provides a very natural basis for deriving useful asymptotic expressions. However, our consideration there was confined to the case $\sigma<d<2\sigma$. In ref. 8, a new analytical technique based on integral transformations with Mittag–Leffler-type kernels was used to derive the finite-size scaling function for the free energy of the mean spherical model subject to the same restrictions: fully finite geometry and $\sigma<d<2\sigma$.

In the present paper the approach suggested in refs. 7 and 8 is developed for the study of the most general case: (a) geometry $L^{d-d'}\times\infty^{d'}$ with periodic boundary conditions imposed along the $d^*=d-d'$ finite-size directions, d and d' being arbitrary real parameters, $0<d'<d$; (b) the

relative magnitudes of d and σ are such that the studied domain includes the upper critical dimension $d = 2\sigma$ as well as the mean-field regime $d > 2\sigma$. This general case is interesting not only from the conceptual point of view, but for practical reasons as well, since logarithmic and mean-field effects violating the standard finite-size scaling[11,12] become observable at the physical dimensionalities $d = 1, 2, 3$, provided a sufficiently small value of σ is realized. Moreover, the investigation for general real values of d and σ sheds light on the use of the ε-expansion in the finite-size scaling theory. For a discussion of the latter problem see refs. 2, 3, 5, and 6.

Let us introduce some concepts and notations. The Hamiltonian of the model is defined on a finite d-dimensional lattice $\mathscr{L}_{d,N} = L_1 \times L_2 \times \cdots L_d$ of $N = \prod_{j=1}^{d} L_j$ sites, subject to periodic boundary conditions so that interactions with repeated images of the system are taken into account.[1,13] As is well known, the exact partition function of the mean spherical model is given by the expression (see, e.g., ref. 13)

$$Z_N(\phi \mid \beta, H) = \pi^{N/2} \exp\left[\frac{\beta^2 H^2 N}{2\beta \hat{J}(\mathbf{0})\, \phi}\right] \prod_{\mathbf{q}} \left\{\frac{1}{2}\, \beta[\hat{J}(\mathbf{0})(\phi + 1) - \hat{J}(\mathbf{q})]\right\}^{-1/2} \tag{1.3}$$

Here

$$\mathbf{q} = \{2\pi p_1/L_1, ..., 2\pi p_d/L_d\}, \qquad p_k = 0, \pm 1, ... \ (\mathrm{mod}\ L_k), \quad k = 1, ..., d$$

and $\hat{J}(\mathbf{q})$ is the Fourier transform of the pair interaction potential, which in the case of long-range interactions decaying at large distances r as $r^{-d-\sigma}$, with $\sigma > 0$, has the long-wavelength asymptotic form

$$\hat{J}(\mathbf{q}) \simeq \hat{J}(\mathbf{0})(1 - \rho_\sigma\, |\mathbf{q}|^\sigma) \tag{1.4}$$

The usual nearest-neighbor interaction corresponds to $\sigma = 2$. In Eq. (1.3), $\beta > 0$ is the inverse temperature, $H \in \mathbb{R}^1$ is an external magnetic field, and the spherical field ϕ obeys the equation

$$\beta \rho_\sigma \hat{J}(\mathbf{0})\{1 - [H/\rho_\sigma \hat{J}(\mathbf{0})\, \tilde{\phi}]^2\} = W_{d,\sigma}^{(L)}(\tilde{\phi}) \tag{1.5}$$

where $\tilde{\phi} = \phi/\rho_\sigma$ and in the long-wavelength approximation

$$W_{d,\sigma}^{(L)}(\tilde{\phi}) = N^{-1} \sum_{\mathbf{q}} (\tilde{\phi} + |\mathbf{q}|^\sigma)^{-1} \tag{1.6}$$

From previous investigations[1,13] it is known that the critical properties of the model in the thermodynamic limit are determined by the asymptotic behavior of the Watson type integral in the long-wavelength approximation,

$$W_{d,\sigma}(\tilde{\phi}) = (2\pi)^{-d} \int_{-\pi}^{\pi} \cdots \int d^d q\, (\tilde{\phi} + |\mathbf{q}|^\sigma)^{-1} \tag{1.7}$$

when $\tilde{\phi} \to 0^+$.

In the theory of critical finite-size scaling it is essential to evaluate the asymptotic behavior of the function (1.6) for large L_j and finite values of the parameter

$$y = \tilde{\phi}^{1/\sigma} L/2\pi, \qquad L = \left(\prod_{j=1}^{d} L_j \right)^{1/d} \tag{1.8}$$

Henceforth the function (1.6) will be referred to as the finite-size Watson function.

2. LARGE-L ASYMPTOTIC FORM OF THE FINITE-SIZE WATSON FUNCTION

The asymptotic form of the finite-size Watson function (1.6) when $L_j \to \infty$ in the block geometry, y finite, has been analyzed in refs. 7 and 8 in the case of $\sigma < d < 2\sigma$, $0 < \sigma \leqslant 2$, $L_j = L$, $j = 1,..., d$. Here we present an approach which allows one to treat in a unified way the general case when the system is infinite in d' dimensions and finite in $d^* = d - d'$ dimensions, including the borderline cases $d = \sigma I$, I being a positive integer. Four steps are taken in this approach.

A. Using the identity

$$(1 + z^\sigma)^{-1} = \int_0^\infty dx\, e^{-z^2 x} x^{\sigma/2 - 1} E_{\sigma/2, \sigma/2}(-x^{\sigma/2}) \tag{2.1}$$

where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function (some useful properties of these functions are summarized in Appendix A of ref. 8)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad (\alpha > 0) \tag{2.2}$$

we obtain for the finite-size Watson function

$$W_{d,\sigma}^{(L)}(\tilde{\phi}) = \tilde{\phi}^{-1} \int_0^\infty dx \left\{ \prod_{j=1}^{d} Q_{L_j}(x \tilde{\phi}^{-2/\sigma}) \right\} x^{\sigma/2 - 1} E_{\sigma/2, \sigma/2}(-x^{\sigma/2}) \tag{2.3}$$

where

$$Q_L(a) = L^{-1} \sum_{n = -(L-1)/2}^{(L-1)/2} \exp\left[-a \left(\frac{2\pi n}{L} \right)^2 \right] \tag{2.4}$$

In the case of $L^{d-d'} \times \infty^{d'}$ geometry of the system, Eq. (2.3) takes the form

$$W_{d,d',\sigma}^{(L)}(\tilde{\phi}) = \frac{\tilde{\phi}^{d'/\sigma - 1}}{(4\pi)^{d'/2}} \int_0^\infty dx\, [Q_L(x\tilde{\phi}^{-2/\sigma})]^{d-d'}$$
$$\times [\Phi(\pi x^{1/2} \tilde{\phi}^{-1/\sigma})]^{d'} x^{(\sigma - d')/2 - 1} E_{\sigma/2, \sigma/2}(-x^{\sigma/2}) \tag{2.5}$$

where $\Phi(\cdot)$ is the error function and we have taken into account that $L_1 = \cdots = L_{d^*} = L$, while the limit $L_j \to \infty$ for $j = d^* + 1,..., d$ was taken, with the result

$$\lim_{L_j \to \infty} Q_{L_j}(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, e^{-ax^2} = (4\pi a)^{-1/2}\, \Phi(\pi a^{1/2})$$

In expression (2.5) the functional dependence on the dimensions d and d' is explicitly shown. It allows one to treat d and d' as real numbers (not just integers).

Step A reduces the problem of evaluating the asymptotic behavior of the $(d^* = d - d')$-dimensional sum (1.6) to the corresponding one-dimensional problem for the sum (2.4).

B. We continue the function $\exp[-a(2\pi n/L)^2]$ from the interval $n \in [-L/2, L/2]$ to the rest of the real line $\mathbb{R}^1$ periodically with respect to the wavenumber $q = 2\pi n/L$, with period 2π. This trick allows us to use the Poisson summation formula for periodic functions[2]

$$L^{-1} \sum_{n=-(L-1)/2}^{(L-1)/2} G\left(\frac{2\pi n}{L}\right) = \sum_{l=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{dq}{2\pi}\, G(q)\, e^{iqlL} \tag{2.6}$$

where $G(q) = G(q + 2\pi)$ for each q. With the aid of Eq. (2.6) we obtain

$$Q_L(a) \approx (4\pi a)^{-1/2} \left[\Phi(\pi a^{1/2}) + \sum_{l=-\infty}^{\infty}{}' \exp(-l^2L^2/4a) \right] \tag{2.7}$$

where the primed summation denotes that the term with $l = 0$ has been omitted. The large-L asymptotic equality (2.7) is justified by the fact that in the exact expression the error function corresponding to the terms with $l \neq 0$ has the argument[7,8]

$$z = \pi a^{1/2} + \tfrac{1}{2} i l L a^{-1/2}$$

and therefore tends to unity exponentially fast when $L \to \infty$.

Next, upon raising both sides of (2.7) to the power d^*, one makes the second approximation

$$[Q_L(a)]^{d^*} \simeq (4\pi a)^{-d^*/2} \left\{ [\Phi(\pi a^{1/2})]^{d^*} + \sum_{l \in \mathbb{Z}^{d^*}}{}' \exp(-l^2L^2/4a) \right\} \tag{2.8}$$

which follows if in all terms of the form

$$[\Phi(\pi a^{1/2})]^m \exp(-l^2L^2/4a) \tag{2.9}$$

with $1 \leqslant m \leqslant d^* - 1$ and $l = |\mathbf{l}| \neq 0$ one replaces the error function $\Phi(\pi a^{1/2})$ by unity. This approximation is legitimate since the exponential function in (2.9) cuts off the contribution from small values of a.

We emphasize that the two approximations made in Eqs. (2.7) and (2.8) are the only approximations connected with the finite but large enough size of the system.

By substituting (2.8) in expression (2.5) we obtain

$$W^{(L)}_{d,d',\sigma}(\tilde{\phi}) = W_{d,\sigma}(\tilde{\phi}) + \delta W_{d,d',\sigma}(\tilde{\phi}) \tag{2.10}$$

where the bulk contribution [see Eq. (1.7)] has been separated from the finite-size one. For the latter we obtain

$$\delta W^{(L)}_{d,d',\sigma}(\tilde{\phi}) = \frac{\tilde{\phi}^{d/\sigma - 1}}{(4\pi)^{d/2}} \sum_{\mathbf{l} \in \mathbb{Z}^{d^*}}' \int_0^\infty dx \exp\left(\frac{-\pi^2 y^2 l^2}{x}\right) \times x^{(\sigma - d)/2 - 1} E_{\sigma/2, \sigma/2}(-x^{\sigma/2}) \tag{2.11}$$

It is convenient to introduce the function

$$F_{d,d',\sigma}(y) = -(2\pi y)^{-\sigma} + \frac{\pi^{d/2} y^{d-\sigma}}{(2\pi)^\sigma} \sum_{\mathbf{l} \in \mathbb{Z}^{d^*}}' \int_0^\infty dx \exp\left(\frac{-\pi^2 y^2 l^2}{x}\right) \times x^{(\sigma - d)/2 - 1} E_{\sigma/2, \sigma/2}(-x^{\sigma/2}) \tag{2.12}$$

with the aid of which Eq. (2.11) may be written in the form

$$\delta W^{(L)}_{d,d',\sigma}(\tilde{\phi}) = (\tilde{\phi} L^d)^{-1} + L^{-d+\sigma} F_{d,d',\sigma}(\tilde{\phi}^{1/\sigma} L/2\pi) \tag{2.13}$$

We note that for $d = 3$, $d' = 0$, and $\sigma = 2$ the function (2.12) simplifies to

$$F_{3,0,2}(y) = \frac{1}{4\pi} \sum_{\mathbf{l} \in \mathbb{Z}^3}' l^{-1} \exp(-2\pi y l) - (2\pi y)^{-2} \tag{2.14}$$

The next two steps are connected with identical transformations of the lattice sum in the right-hand side of Eq. (2.12). The identities thus obtained generalize the following identity due to Chaba and Pathria[14]:

$$\sum_{\mathbf{l} \in \mathbb{Z}^3}' l^{-1} \exp(-2\pi y l) - \frac{1}{\pi y^2} = \frac{C_3}{\pi} + 2\pi y - \frac{y^2}{\pi} \sum_{\mathbf{l} \in \mathbb{Z}^3} l^{-2} (l^2 + y^2)^{-1}$$

where C_3 is a universal constant. The representation given by the right-hand side of the above equation is especially convenient for small values of y.

This step makes use of the Poisson summation formula in order to separate the bulk contribution from the finite-size one. The approximations of large L are localized in the derivation of the asymptotic equalities (2.7) and (2.8). As a result of these approximations the whole dependence on the finite size L of the system enters through the characteristic combination $y = \tilde{\phi}^{1/\sigma} L/2\pi$ [see Eq. (2.11)].

C. The application of the Jacobi identity

$$A(u) = \left(\frac{\pi}{u}\right)^{1/2} A\left(\frac{\pi^2}{u}\right) \tag{2.15}$$

where

$$A(u) = \sum_{l=-\infty}^{\infty} e^{-ul^2} \tag{2.16}$$

to the sum in the right-hand side of Eq. (2.12) yields the result

$$\begin{aligned} F_{d,d',\sigma}(y) = {} & D^{(0)}_{d',\sigma}(2\pi y)^{d'-\sigma} - (2\pi y)^{-\sigma} \\ & + \frac{\pi^{d'/2}}{(2\pi)^\sigma} \int_0^\infty du\, u^{(\sigma-d')/2-1} E_{\sigma/2,\sigma/2}(-y^\sigma u^{\sigma/2}) \\ & \times \left[A^{d^*}(u) - \left(\frac{\pi}{u}\right)^{d^*/2} - 1 \right] \end{aligned} \tag{2.17}$$

where (for $0 < d' < \sigma$)

$$D^{(0)}_{d',\sigma} = 2\pi \left[(4\pi)^{d'/2} \Gamma\left(\frac{d'}{2}\right) \sigma \sin \frac{d'\pi}{\sigma} \right]^{-1} \tag{2.18}$$

For $d = 4$, $d' = 0$, and $\sigma = 2$, when $E_{1,1}(z) = \exp(z)$, Eq. (2.17) reduces to the function $F_1(y)$ introduced earlier by a number of authors in finite-size scaling theory.[3,15–17]

In step C we make use of the Poisson identity (2.15) in order to transform the representation (2.12) of the finite-size contribution into another form, Eq. (2.17), which is more convenient for small y.

We note that an alternative approach, avoiding the use of the Jacobi identity, has been suggested in ref. 8.

D. The identical transformations in this step separate explicitly the first few leading-order terms in $y \to 0^+$ at different ratios of d and σ. For the reader's convenience all the calculations are given in the Appendix. Here we present the final expressions only.

In the case when $0 < d' < \sigma$, $\sigma I < d < \sigma(I+1)$,

$$F_{d,d',\sigma}(y) = -(2\pi y)^{-\sigma} + D^{(0)}_{d',\sigma}(2\pi y)^{d'-\sigma} - D^{(0)}_{d,\sigma}(2\pi y)^{d-\sigma} + \sum_{k=1}^{I} (-1)^{k+1} C_{d,d',\sigma k}[(2\pi)^{\sigma k} \Gamma(\sigma k/2)]^{-1} (2\pi y)^{\sigma(k-1)} + \pi^{d'/2}(2\pi)^{-\sigma} \sum_{\mathbf{n} \in \mathbb{Z}^{d^*}}' n^{-\sigma+d'} v^{(I)}_{\sigma/2,d'/2}(y^2/n^2) \tag{2.19}$$

where

$$C_{d,d',\sigma k} = \pi^{d'/2} \int_0^\infty du\, u^{(\sigma k - d')/2-1} \left[A^{d^*}(u) - \left(\frac{\pi}{u}\right)^{d^*/2} - 1 \right] \tag{2.20}$$

converges for $d' < \sigma k < d$, and

$$v^{(I)}_{\alpha,\beta}(t) = \int_0^\infty dx\, x^{\alpha-\beta-1} e^{-x} \left\{ E_{\alpha,\alpha}[-(xt)^\alpha] - \sum_{k=0}^{I-1} \frac{(-1)^k (xt)^{\alpha k}}{\Gamma[\alpha(k+1)]} \right\} \tag{2.21}$$

Note that the last term in the right-hand side of Eq. (2.19) is $\mathcal{O}(y^{\sigma I})$. In the important special case of fully finite geometry, $d' = 0$, $d^* = d$, by using Eqs. (2.1) and (2.21) (with $\beta = 0$) and taking the limit $d' \to 0$ in $D^{(0)}_{d',\sigma}$, we obtain

$$F_{d,0,\sigma}(y) = \sum_{k=1}^{I} (-1)^{k+1} C_{d,0,\sigma k}[(2\pi)^{\sigma k} \Gamma(\sigma k/2)]^{-1} (2\pi y)^{\sigma(k-1)} - D^{(0)}_{d,\sigma}(2\pi y)^{d-\sigma} + (-1)^I y^{\sigma I}(2\pi)^{-\sigma} \sum_{l \in \mathbb{Z}^d}' l^{-\sigma I}(l^\sigma + y^\sigma)^{-1} \tag{2.22}$$

From a physical point of view the given parameters of the problem are the space dimensionality d and the interaction exponent σ. If the ratio d/σ is nonintegral, then the identity (2.19) is to be used with $I = [d/\sigma]$, where the symbol $[x]$ denotes the entire part of x. If $d = I\sigma$, where $I \geqslant 2$ is an integer, then expression (2.19) for $F_{d,d',\sigma}(y)$ has to be modified in the following way:

$$F_{\sigma I,d',\sigma}(y) = -(2\pi y)^{-\sigma} + D^{(0)}_{d'.\sigma}(2\pi y)^{d'-\sigma} + \tilde{D}^{(I)}_{I\sigma,d',\sigma}(2\pi y)^{\sigma(I-1)} + \sum_{k=1}^{I-1} (-1)^{k+1} C_{d,d',\sigma k}[(2\pi)^{\sigma k} \Gamma(\sigma k/2)]^{-1} (2\pi y)^{\sigma(k-1)} - 2(-1)^I[\sigma(4\pi)^{\sigma I/2} \Gamma(\sigma I/2)]^{-1} (2\pi y)^{\sigma(I-1)} \ln(2\pi y)^\sigma + \pi^{d'/2}(2\pi)^{-\sigma} \sum_{\mathbf{n} \in \mathbb{Z}^{d^*}}' n^{-\sigma+d'} v^{(I)}_{\sigma/2,d'/2}(y^2/n^2) \tag{2.23}$$

where the new constant $\tilde{D}^{(I)}_{d,d',\sigma}$ is given by

$$\tilde{D}^{(I)}_{I\sigma,d',\sigma} = -(4\pi)^{-\sigma I/2} \int_0^\infty \frac{du}{u} \left\{ u^{-\sigma(I-1)/2} \left[E_{\sigma/2,\sigma/2}(-u^{\sigma/2}) - \sum_{k=0}^{I-2} \frac{(-1)^k u^{\sigma k/2}}{\Gamma[\sigma(k+1)/2]} \right] + \frac{(-1)^I}{\Gamma(\sigma I/2)} \left(\frac{u}{\pi}\right)^{(\sigma I - d')/2} [A^{\sigma I - d'}(u) - 1] \right\} \tag{2.24}$$

Of definite interest for the finite-size scaling theory is the case $d = \sigma$, which corresponds to the lower critical dimension. In this case

$$F_{\sigma,d',\sigma}(y) = -(2\pi y)^{-\sigma} + D^{(0)}_{d',\sigma}(2\pi y)^{d'-\sigma} + \tilde{D}^{(1)}_{\sigma,d',\sigma} + 2[\sigma(4\pi)^{\sigma/2}\Gamma(\sigma/2)]^{-1}\ln(2\pi y)^\sigma + \pi^{d'/2}(2\pi)^{-\sigma} \sum_{\mathbf{n}\in\mathbb{Z}^{d^*}}{}' n^{-\sigma+d'} v^{(1)}_{\sigma/2,d'/2}(y^2/n^2) \tag{2.25}$$

where

$$\tilde{D}^{(1)}_{\sigma,d',\sigma} = -(4\pi)^{-\sigma/2} \int_0^\infty \frac{du}{u} \left\{ E_{\sigma/2,\sigma/2}(-u^{\sigma/2}) - \frac{1}{\Gamma(\sigma/2)} \left(\frac{u}{\pi}\right)^{(\sigma-d')/2} [A^{\sigma-d'}(u) - 1] \right\} \tag{2.26}$$

Note that by using identical transformations in the last two steps we have obtained representations for the function $F_{d,d',\sigma}(y)$ generalizing the results of ref. 1. Our results hold for all $d/\sigma \in [1, \infty)$, $0 < \sigma \leqslant 2$, $0 \leqslant d' < \sigma$: Eq. (2.19) holds for $I < d/\sigma < I+1$ ($I = 1, 2,\ldots$), Eq. (2.23) for $d/\sigma = I \geqslant 2$, and Eq. (2.25) for $d/\sigma = 1$.

The representations of the function $F_{d,d',\sigma}(y)$ obtained in this section form the basis of our further investigation of finite-size effects.

3. CRITICAL FINITE-SIZE SCALING

According to the finite-size hypothesis,[11,12] in a close neighborhood of the critical temperature T_c of the infinite system, the rounding and shifting of the singularities in the thermodynamic functions is controlled by the scaled variable $x = L/\xi_\infty$, where ξ_∞ is the bulk correlation length. These arguments have to be modified, however, in the case of $\mathcal{O}(n)$ models with $n \geqslant 2$ below T_c, when, in the absence of symmetry-breaking fields, $\xi_\infty = \infty$ for all $T < T_c$.[1,18] That is why we first consider the case $T > T_c$.

The representations of the function $F_{d,d',\sigma}(y)$ obtained in Section 2 [see Eqs. (2.19), (2.23), and (2.25)] allow for a direct testing of the finite-size scaling hypothesis[11,12] for different relative values of d, d', and σ. Denote by $\tilde{\phi}_L$ the solution of the finite-size equation for the spherical field which follows from Eqs. (1.5), (2.10), and (2.13),

$$\begin{aligned} &\rho_\sigma \beta \hat{J}(\mathbf{0})\{1-[H/\rho_\sigma \hat{J}(\mathbf{0})\,\tilde{\phi}]^2\} \\ &\quad = W_{d,\sigma}(\tilde{\phi}) + L^{-d}\tilde{\phi}^{-1} + L^{-d+\sigma}F_{d,d',\sigma}(\tilde{\phi}^{1/\sigma}L/2\pi) \end{aligned} \tag{3.1}$$

and let $\tilde{\phi}_\infty$ be the solution of the corresponding bulk equation which follows from (3.1) in the limit $L \to \infty$. Define then effective correlation lengths $\xi_L = \tilde{\phi}_L^{-1/\sigma}$ and $\xi_\infty = \tilde{\phi}_\infty^{-1/\sigma}$ for the finite and infinite systems, respectively. Note that these definitions differ from the moment-based definitions of Brezín[2] and Shapiro and Rudnick.[6] The reason is that in the case of long-range interactions the latter definitions fail in the bulk limit since the second moment of the pair correlation function diverges when $L \to \infty$. We use the definition of an effective correlation length which can be motivated by the asymptotic form of the large-distance finite-size pair correlation function.[19] In the case of a fully finite system and short-range interactions there is a simple correspondence between the different definitions of ξ_L; see Eq. (3.7) below.

In terms of the correlation lengths, Eq. (3.1) may be rewritten in the form ($H=0$)

$$W_{d,\sigma}(\xi_\infty^{-\sigma}) - W_{d,\sigma}(\xi_L^{-\sigma}) = L^{-d+\sigma}\{(\xi_L/L)^\sigma + F_{d,d',\sigma}(L/2\pi\xi_L)\} \tag{3.2}$$

If the finite-size hypothesis holds from (3.2), it would follow that[2]

$$\xi_L/L = X(\xi_\infty/L) \tag{3.3}$$

with a universal scaling function $X(\cdot)$.

Let us consider now the different cases which correspond to different relative magnitudes of d and σ, assuming that $d' < \sigma$.

(i) $d = \sigma$. This case corresponds to the lower critical dimension of the model. The bulk thermodynamic functions exhibit essential singularities at the critical point $T_c = 0$. The Watson-type integral (1.7) now has the small-$\tilde{\phi}$ asymptotic form[20]

$$W_{\sigma,\sigma}(\tilde{\phi}) \simeq -2[\sigma(4\pi)^{\sigma/2}\,\Gamma(\sigma/2)]^{-1} \ln \tilde{\phi} \tag{3.4}$$

and Eq. (3.2) takes the finite-size scaling form

$$-2[(4\pi)^{\sigma/2}\,\Gamma(\sigma/2)]^{-1} \ln(\xi_L/\xi_\infty) = (\xi_L/L)^\sigma + F_{\sigma,d',\sigma}(L/2\pi\xi_L) \tag{3.5}$$

Equation (3.5) is equivalent to Eq. (3.3) with a scaling function depending on σ and d'. When $L/\xi_L \ll 1$, by using the leading-order behavior of $F_{\sigma,d',\sigma}(y)$ [see (2.25)], we obtain the solution of Eq. (3.5) in an explicit form:

$$\xi_L/L = \{2[(4\pi)^{\sigma/2}\,\Gamma(\sigma/2)\,D^{(0)}_{d',\sigma}]^{-1}\ln(\xi_\infty/L)\}^{1/(\sigma-d')} \tag{3.6}$$

To compare with the results of Shapiro and Rudnick[6] for the case of fully finite geometry ($d'=0$) and nearest neighbor interaction ($\sigma=2$), we have to take into account the relationship between their correlation length ξ_L^{SR} and the one accepted here, ξ_L, namely

$$(\xi_L^{\text{SR}}/L)^2 = \text{const}\cdot[1+(1/2\pi)(\xi_L/L)^{-2}]^{-1} \tag{3.7}$$

By substituting (3.6) into (3.7), we obtain

$$(\xi_L^{\text{SR}}/L)^2 \simeq \text{const}\cdot[1-1/\ln(\xi_\infty/L)] \tag{3.8}$$

in full agreement with ref. 6.

(ii) $\sigma < d < 2\sigma$. In this case, when $T\to T_c^+$ one obtains the asymptotic expansion[20]

$$W_{d,\sigma}(\bar{\phi}) \simeq W_{d,\sigma}(0) - |D^{(1)}_{d,\sigma}|\,\bar{\phi}^{(d-\sigma)/\sigma} \tag{3.9}$$

Now Eq. (3.1) becomes ($H=0$)

$$|D^{(1)}_{d,\sigma}|\,[(L/\xi_L)^{d-\sigma}-(L/\xi_\infty)^{d-\sigma}] = (\xi_L/L)^\sigma + F_{d,d',\sigma}(L/2\pi\xi_L) \tag{3.10}$$

thus verifying the validity of the finite-size scaling form (3.3) with a scaling function $X(\cdot)$ depending on d, d', and σ.

Note that the coefficient $D^{(1)}_{d,\sigma}$ in the left-hand side of Eq. (3.10) becomes singular when $d\to\sigma^+$ or $d\to 2\sigma^-$, as is seen from Eq. (A.4). In these cases, however, one should return to the initial equations (3.1) or (3.2), respectively, and take into account the following.

(a) When $d\to\sigma^+$, both constants $W_{d,\sigma}(0)$ and $D^{(1)}_{d,\sigma}$ in Eq. (3.9) diverge,

$$\begin{aligned} W_{d,\sigma}(0) &= 2[(4\pi)^{\sigma/2}\,\Gamma(\sigma/2)\,\varepsilon]^{-1} + \mathcal{O}(1) \\ |D^{(1)}_{d,\sigma}| &= 2[(4\pi)^{\sigma/2}\,\Gamma(\sigma/2)\,\varepsilon]^{-1} + \mathcal{O}(1) \end{aligned} \tag{3.11}$$

but the divergent terms cancel out and the result is

$$W_{\sigma+\varepsilon,\sigma}(\bar{\phi}) = -2[(4\pi)^{\sigma/2}\,\Gamma(\sigma/2)]^{-1}\,[\ln\bar{\phi}^{1/\sigma}+\mathcal{O}(1)] + \mathcal{O}(\varepsilon\ln\bar{\phi}) \tag{3.12}$$

The right-hand side of Eq. (3.10) is continuous in the limit $d \to \sigma^+$ and the function $F_{d,d',\sigma}(y)$ takes the form (2.25) in that limit. Thus, we obtain Eq. (3.5) as the correct $d \to \sigma^+$ limit of the equation for the spherical field (3.1) at $H=0$.

(b) When $d \to 2\sigma^-$, the bulk term again has to be reevaluated starting from the qualitatively correct approximation[21]

$$W_{d,\sigma}(\tilde{\phi}) \simeq W_{d,\sigma}(0) - 2\tilde{\phi}^{(d-\sigma)/\sigma}[(4\pi)^{d/2}\,\Gamma(d/2)]^{-1} \times \int_0^{q_0\tilde{\phi}^{-1/\sigma}} dx\, x^{d-\sigma-1}(1+x^\sigma)^{-1} \tag{3.13}$$

where q_0 is the radius of the effective spherical Brillouin zone. The result is

$$W_{2\sigma-\varepsilon,\sigma}(\tilde{\phi}) = W_{2\sigma,\sigma}(0) + 2[\sigma(4\pi)^\sigma\,\Gamma(\sigma)]^{-1}\,\tilde{\phi}\ln\tilde{\phi} + \mathcal{O}(\varepsilon\ln\tilde{\phi}) \tag{3.14}$$

and we are led to the consideration of the next case.

(iii) $d=2\sigma$. After passing to the limit $d \to 2\sigma^-$ in the right-hand side of Eq. (3.2), which leads to the representation (2.23) with $I=2$ for the function $F_{d,d',\sigma}(y)$, and taking into account (3.14), we obtain the equation

$$\begin{aligned} 2[(4\pi)^\sigma\,\Gamma(\sigma)]^{-1}\,&[(L/\xi_\infty)^\sigma \ln(L/\xi_\infty) - (L/\xi_L)^\sigma \ln(L/\xi_L)] \\ &- 2\ln L[(4\pi)^\sigma\,\Gamma(\sigma)]^{-1}\,[(L/\xi_\infty)^\sigma - (L/\xi_L)^\sigma] \\ &= (\xi_L/L)^\sigma + F_{2\sigma,d',\sigma}(L/2\pi\xi_L) \end{aligned} \tag{3.15}$$

The presence of the second term, proportional to $\ln L$, in the left-hand side of Eq. (3.15) violates the finite-size scaling relationship (3.3). Instead, following the idea of Luck,[5] one may derive a modified finite-size scaling relationship. To this end, we introduce the new variables

$$\begin{aligned} x &= (L/\xi_\infty)(\ln L)^{1/(2\sigma-d')} \\ z &= (\xi_L/L)(\ln L)^{-1/(2\sigma-d')} \end{aligned} \tag{3.16}$$

and, ignoring terms of $\mathcal{O}(\ln\ln L/\ln L)$, rewrite Eq. (3.15) in the form

$$\tfrac{1}{2}(4\pi)^\sigma\,\Gamma(\sigma)\,D^{(0)}_{d',\sigma} z^{2\sigma-d'} + (xz)^\sigma\,(1-\ln x/\ln L) - 1 = 0 \tag{3.17}$$

At $T \neq T_c$ and large enough L the second term in the brackets may be ignored, too, and then the relationship between the variables x and z resembles the finite-size scaling equation (3.3). At $T=T_c$ from (3.17) we obtain

$$\xi_L = [\tfrac{1}{2}(4\pi)^\sigma\,\Gamma(\sigma)\,D^{(0)}_{d',\sigma}]^{-1/(2\sigma-d')}\,L(\ln L)^{1/(2\sigma-d')} \tag{3.18}$$

In the particular case $d'=1$, $\sigma=2$, Eq. (3.17) was obtained in ref. 5.

(iv) $d>2\sigma$. In this case, when $T\to T_c$,[20]

$$W_{d,\sigma}(\bar{\phi}) \simeq W_{d,\sigma}(0) - |W'_{d,\sigma}(0)|\,\bar{\phi} \tag{3.19}$$

and Eq. (3.2) takes the form

$$|W'_{d,\sigma}(0)|\,(\xi_L^{-\sigma} - \xi_\infty^{-\sigma}) = L^{-d+\sigma}(\xi_L/L)^\sigma + L^{-d+\sigma} F_{d,d',\sigma}(L/2\pi\xi_L) \tag{3.20}$$

At $L/\xi_L \ll 1$ from the representation (2.23) with $I>2$ we obtain

$$F_{\sigma I,d',\sigma}(L/2\pi\xi_L) \simeq D^{(0)}_{d',\sigma}(L/\xi_L)^{-\sigma+d'} - (L/\xi_L)^{-\sigma} \tag{3.21}$$

Introducing the variables

$$\begin{aligned} \xi_L/L &= zL^{(d-2\sigma)/(2\sigma-d')} \\ L/\xi_\infty &= xL^{-(d-2\sigma)/(2\sigma-d')} \end{aligned} \tag{3.22}$$

and taking into account (3.21), we rewrite Eq. (3.20) in the form

$$D^{(0)}_{d',\sigma}\,|W'_{d,\sigma}(0)|^{-1}\, z^{2\sigma-d'} + (xz)^\sigma - 1 = 0 \tag{3.23}$$

which is similar to the case $d=2\sigma$. Such a modified finite-size scaling has been suggested in ref. 5. At $T=T_c$ from Eqs. (3.22) and (3.23) we obtain

$$\xi_L = [\,|W'_{d,\sigma}(0)|/D^{(0)}_{d',\sigma}]^{1/(2\sigma-d')}\, L^{(d-d')/(2\sigma-d')} \tag{3.24}$$

In the particular case $\sigma=2$ the relationship (3.24) reduces to the results obtained by Singh and Pathria.[22]

4. FINITE-SIZE SCALING AT FIRST-ORDER PHASE TRANSITIONS

As already mentioned, the representations obtained in Section 2 for the function $F_{d,d',\sigma}(y)$ are especially advantageous when $y\to 0^+$, which is the case of $T<T_c$, $d'<\sigma$, and $L\to\infty$. Let $\sigma I<d<\sigma(I+1)$. With the use of (2.19), taking into account that in the presence of an external magnetic field H the magnetization per spin in the finite system is

$$m_L(T, H) = H/\rho_\sigma \hat{J}(\mathbf{0})\,\bar{\phi}_L \tag{4.1}$$

we may write the equation of state [cf. Eq. (1.5)]

$$W^{(L)}_{d,d',\sigma}(\bar{\phi}) = \rho_\sigma \beta \hat{J}(\mathbf{0})\{1 - [H/\rho_\sigma \hat{J}(\mathbf{0})\,\bar{\phi}]^2\} \tag{4.2}$$

in the form

$$\frac{1}{2\mathscr{K}}\left[W_{d,\sigma}(\bar{\phi})-W_{d,\sigma}(0)\right]+\frac{1}{2\mathscr{K}L^{d-\sigma}}\left[D^{(0)}_{d',\sigma}L^{d'-\sigma}\left(\frac{1}{2\mathscr{K}}\frac{h}{m}\right)^{(d'-\sigma)/\sigma}\right.$$
$$\left.-D^{(I)}_{d,\sigma}L^{d-\sigma}\left(\frac{1}{2\mathscr{K}}\frac{h}{m}\right)^{(d-\sigma)/\sigma}-\frac{1}{\mathscr{K}}Y^{(s)}_{d,d'.\sigma}\left(\frac{L^{\sigma}}{2\mathscr{K}}\frac{h}{m}\right)\right]$$
$$=m_0^2(T)-m^2 \tag{4.3}$$

Here

$$\mathscr{K}=\tfrac{1}{2}\rho_{\sigma}\beta\hat{J}(\mathbf{0}),\qquad h=\beta H \tag{4.4}$$

and we have introduced the "spin-wave" scaling function[1]

$$Y^{(s)}_{d,d',\sigma}(z)=\frac{1}{2}\sum_{k=1}^{I}(-1)^k C_{d,d',\sigma k}\left[(2\pi)^{\sigma k}\Gamma\left(\frac{\sigma k}{2}\right)\right]^{-1}z^{(k-1)}$$
$$-\frac{1}{2}\pi^{d'/2}(2\pi)^{-\sigma}\sum_{\mathbf{n}\in\mathbb{Z}^{d^*}}{}' n^{-\sigma+d'}v^{(I)}_{\sigma/2,d'/2}\left(\frac{z^{2/\sigma}}{4\pi^2n^2}\right) \tag{4.5}$$

which generalizes expression (3.25) of ref. 1 to the case of real values of d', $0<d'<\sigma$. Note that we substitute in Eq. (4.3) the expansion[1]

$$W_{d,\sigma}(\bar{\phi})=W_{d,\sigma}(0)+\sum_{k=1}^{I-1}(-1)^k b_k\bar{\phi}^k+D^{(I)}_{d,\sigma}\bar{\phi}^{(d-\sigma)/\sigma} \tag{4.6}$$

and finally obtain

$$m_0^2(T)-m^2=\frac{D^{(0)}_{d',\sigma}}{2\mathscr{K}}L^{-d^*}\left(\frac{1}{2\mathscr{K}}\frac{h}{m}\right)^{(d'-\sigma)/\sigma}$$
$$+\frac{1}{2\mathscr{K}}\sum_{k=1}^{I-1}\bar{b}_k\left(\frac{h}{m}\right)^k-\frac{1}{\mathscr{K}L^{d-\sigma}}Y^{(s)}_{d,d',\sigma}\left(\frac{L^{\sigma}}{2\mathscr{K}}\frac{h}{m}\right) \tag{4.7}$$

where $\bar{b}_k=(-2\mathscr{K})^{-k}b_k$.

At $d'=0$, Eq. (4.7) reduces to the asymptotic equation of state (3.26) of ref. 1.

Note that in the case of an infinite cylinder ($d'=1$), Eq. (4.7) corrects Eq. (4.20) of ref. 1 with respect to the term

$$(\bar{\phi}L^d)^{-1}=2\mathscr{K}m/L^dh \tag{4.8}$$

The term (4.8) is the $q=0$ contribution of the sum (1.6) in the case of fully finite geometry, but it should not appear in the case when the system is infinite in $d' > 0$ dimensions, as follows from our representations (2.13) and (2.17).

Taking the limit $h \to 0$ in Eq. (4.7), we easily obtain for the zero-field susceptibility below T_c the leading-order result

$$\chi_0 = \beta \lim_{h \to 0} (m/h) = \beta(2\mathscr{K})^{d'/(\sigma - d')} [m_0^2(T) L^{d^*}/D^{(0)}_{d',\sigma}]^{\sigma/(\sigma - d')} \tag{4.9}$$

which at $d' = 1$ reduces to Eq. (4.21) of ref. 1.

Finally, we note that in the $L^{d^*} \times \infty^{d'}$ geometry the scaled field variable y_A due to Fisher and Privman[1] should be generalized to take the form (see also ref. 19)

$$\begin{aligned} y_A &= hm_0(T) L^d \xi_L^{d'}(T, 0) \\ &= m_0(T) hL^{d^*\sigma/(\sigma - d')}[2\mathscr{K} m_0(T)/D^{(0)}_{d',\sigma}]^{d'/(\sigma - d')} \end{aligned} \tag{4.10}$$

In terms of y_A the leading-order finite-size scaling equation for the magnetization below T_c becomes[19]

$$m_L(T, H) = m_0(T) Y_\infty(y_A) \tag{4.11}$$

where the scaling function Y_∞ obeys the universal equation

$$Y_\infty/(1 - Y_\infty^2)^{\sigma/(\sigma - d')} = y_A/D^{(0)}_{d',\sigma} \tag{4.12}$$

At $d' = 1$ this equation reduces to Eq. (4.27) of ref. 1.

5. DISCUSSION

Here we have suggested a general method for the derivation of finite-size effects in systems with long-range power-law interactions, which also includes as a particular case short-range interactions. To illustrate the method, we have given the asymptotic analysis in the framework of the mean spherical model, but it is clear that there is no difficulty in principle in extending the ideas to the case of more complicated $\mathcal{O}(n)$ lattice or field models.

We have studied systems of general $L^{d-d'} \times \infty^{d'}$ geometry and arbitrary interaction decay exponent $\sigma > 0$, paying special attention to the complications arising at the borderline dimensionalities. The results obtained here shed light on the question of whether the ε-expansion about these dimensionalities is applicable or not. The problem has been discussed with the example of the exactly solvable spherical model by a number of authors.[2,4–6] In this case, due to the fact that the exact form of the equation for the correlation length ξ_L is known at arbitrary temperatures T, one can see that critical finite-size scaling holds only in the close neighborhood of the critical temperature T_c. The solution of the approximate finite-size scaling equation (3.10) exhibits near the upper critical dimension a characteristic singularity in $\varepsilon \to 0^+$:

$$\xi_L(T_c, 0)/L = [\tfrac{1}{2}(4\pi)^\sigma \Gamma(\sigma) D^{(0)}_{d',\sigma} \varepsilon]^{-1/(2\sigma - d')}$$

which has been the subject of discussion.[2,4–6] Our proposition is that this singularity is an artifact of the approximation—the price one has to pay for obtaining a universal finite-size scaling equation. If we retain in the equation the terms which are asymptotically negligible at a fixed dimensionality between the borderline ones, but become singular in ε when the dimensionality approaches the upper or lower critical one, then no ε-singularity appears in the solution, but rather the equation itself is modified, losing its universal form, and so does the solution. For example, at the upper critical dimensionality the ε^{-1} singularity in the correlation length is replaced by $\ln L$ [see Eq. (3.18)].

The above considerations do not contradict, of course, the general idea connecting the breakdown of standard finite-size scaling with the failure of hyperscaling above the upper critical dimension. Comments on this situation with reference to the ε-expansion are given in ref. 6, but the introduction there of the so-called soft spherical constraint, which brings into the theory the dangerous irrelevant variable, shifts the problem out of the framework of the mean spherical model.

The equation for the correlation length (3.2), together with the representations of the function $F_{d,d',\sigma}(y)$ obtained for different relative magnitudes of d and σ, exhibit in a very transparent way the mathematical mechanism of the appearance of different finite-size scaling forms, at least in the case of the spherical model.

The representations given by Eqs. (2.19), (2.23), and (2.25) lead to the Fisher–Privman form[1] of the equation of state for all $d \geqslant \sigma$ and $0 \leqslant d' < \sigma$. They also provide the natural small-y expansion of the function $F_{d,d',\sigma}(y)$, which is especially important below T_c. This has been demonstrated by the expression (4.9) for the susceptibility of the mean spherical model in the region of first-order phase transition.

The representations of the function $F_{d,d',\sigma}(y)$ obtained here are useful in the study of more complicated models, too. This function enters the expressions for the shift of the critical temperature and the renormalization of the interaction constant in the study of the finite-size effects in the φ^4-theory.[3] The same function enters the expressions for the renormalized parameters in the dynamical finite-size scaling theory.[15–17]

We note also that the suggested method can be combined with the

Ewald summation technique used in finite-size scaling by Shapiro and Rudnick.[6] As a result, computationally superior expressions will be obtained for systems with long-range interactions.

APPENDIX

In order to transform Eq. (2.17) into the required form, we subtract from the function $E_{\sigma/2,\sigma/2}(-u^{\sigma/2}y^{\sigma})$ the first $I-1$ terms (the integer I will be defined below) of its expansion (2.2) and then add them in an appropriate way, thus obtaining identically

$$\begin{aligned}
F_{d,d',\sigma}(y) = {} & \pi^{d'/2}(2\pi)^{-\sigma}\int_0^{\infty} du\, u^{(\sigma-d')/2-1}[A^{d^*}(u)-1] \\
& \times\left[E_{\sigma/2,\sigma/2}(-y^{\sigma}u^{\sigma/2})-\sum_{k=0}^{I-1}\frac{(-1)^k\, y^{\sigma k}u^{\sigma k/2}}{\Gamma[\sigma(k+1)/2}\right] \\
& -\pi^{d/2}(2\pi)^{-\sigma}\int_0^{\infty} du\, u^{(\sigma-d)/2-1} \\
& \times\left[E_{\sigma/2,\sigma/2}(-y^{\sigma}u^{\sigma/2})-\sum_{k=0}^{I-1}\frac{(-1)^k\, y^{\sigma k}u^{\sigma k/2}}{\Gamma[\sigma(k+1)/2]}\right] \\
& +\sum_{k=0}^{I-1}\frac{(-1)^k\, y^{\sigma k}}{\Gamma[\sigma(k+1)/2]}\frac{\pi^{d'/2}}{(2\pi)^{\sigma}}\int_0^{\infty} du\, u^{[\sigma(k+1)-d']/2-1} \\
& \times\left[A^{d^*}(u)-\left(\frac{\pi}{u}\right)^{d^*/2}-1\right] \\
& +D^{(0)}_{d',\sigma}(2\pi y)^{d'-\sigma}-(2\pi y)^{-\sigma}
\end{aligned} \tag{A.1}$$

By using the explicit form of $A(u)$ [see Eq. (2.16)] and introducing the function $v^{(I)}_{\alpha,\beta}(t)$ [see (2.21)], we can write the first term in the right-hand side of Eq. (A.1) as

$$\frac{\pi^{d'/2}}{(2\pi)^{\sigma}}\sum_{\mathbf{n}\in\mathbb{Z}^{d^*}}{}' n^{-\sigma+d'}v^{(I)}_{\sigma/2,d'/2}\left(\frac{y^2}{n^2}\right) \tag{A.2}$$

We evaluate the second integral in the right-hand side of Eq. (A.1) exactly with the aid of the identity

$$u^{-d/2}=[\Gamma(d/2)]^{-1}\int_0^{\infty} dt\, t^{d/2-1}e^{-ut} \tag{A.3}$$

which leads to a term proportional to

$$D_{d,\sigma}^{(I)} = \left[(4\pi)^{d/2} \Gamma\left(\frac{d}{2}\right) \right]^{-1} \int_0^\infty dt\, t^{d/2-1} \int_0^\infty du\, u^{\sigma/2-1} e^{-ut} \times \left[E_{\sigma/2,\sigma/2}(-u^{\sigma/2}) - \sum_{k=0}^{I-1} \frac{(-1)^k u^{\sigma k/2}}{\Gamma[\sigma(k+1)/2]} \right]$$

$$= \left[(4\pi)^{d/2} \Gamma\left(\frac{d}{2}\right) \right]^{-1} \int_0^\infty dt\, t^{d/2-1} \times \left[(1+t^{\sigma/2})^{-1} - \sum_{k=0}^{I-1} (-1)^k t^{-\sigma(k+1)/2} \right]$$

$$= 2\pi(-1)^I \left[(4\pi)^{d/2} \Gamma\left(\frac{d}{2}\right) \sigma \sin \frac{(d-\sigma I)\pi}{\sigma} \right]^{-1} = D_{d,\sigma}^{(0)} \qquad \text{(A.4)}$$

In deriving the second of the equalities (A.4), we have used the property (2.2) of the Mittag–Leffler function, and in deriving the third one, use has been made of the elementary identity

$$(1+x)^{-1} - \sum_{k=0}^{N-1} (-1)^k x^k = (-1)^N x^N (1+x)^{-1} \qquad \text{(A.5)}$$

with an obvious substitution for x. Thus, we obtain for this term

$$(2\pi y)^{d-\sigma} D_{d,\sigma}^{(0)}, \qquad d \neq \sigma I, \quad I = \text{integer} \qquad \text{(A.6)}$$

We write the third term in the right-hand side of Eq. (A.1) in the form

$$\sum_{k=1}^{I} (-1)^{k+1} \frac{C_{d,d',\sigma k}}{(2\pi)^{\sigma k} \Gamma(\sigma k/2)} \qquad \text{(A.7)}$$

where $C_{d,d',\sigma k}$ is given by expression (2.20).

By using Eqs. (A.2), (A.4), and (A.17), we can cast the representation (A.1) of $F_{d,d',\sigma}(y)$ into the form (2.19), which is valid for $0 \leqslant d' < \sigma$, $\sigma I < d < \sigma(I+1)$.

In the case $d = \sigma I$, $I \geqslant 2$, the expression for $F_{d,d',\sigma}(y)$ has to be modified in the following way.

Upon adding up the third term in the right-hand side of Eq. (2.19) with the $k=1$ term of the finite sum over $k = 1,\ldots, I$ and the right-hand side of the identity[8]

$$\ln(2\pi y)^\sigma = \frac{\sigma}{2} \int_0^\infty \frac{du}{u} \left[E_{\sigma/2,1}(-u^{\sigma/2}) - E_{\sigma/2,1}(-y^\sigma u^{\sigma/2}) \right] \qquad \text{(A.8)}$$

multiplied beforehand by the coefficient

$$2(-1)^I [\sigma(4\pi)^{\sigma I/2} \Gamma(\sigma I/2)]^{-1} (2\pi y)^{\sigma(I-1)} \tag{A.9}$$

we obtain an expression in which we may set $d=\sigma I$:

$$\begin{aligned} &-(2\pi y)^{\sigma(I-1)} (4\pi)^{-\sigma I/2} \int_0^\infty \frac{du}{u} \Bigg\{ u^{-\sigma(I-1)/2} \Bigg[E_{\sigma/2,\sigma/2}(-u^{\sigma/2}) \\ &\quad - \sum_{k=0}^{I-2} \frac{(-1)^k u^{\sigma k/2}}{\Gamma[\sigma(k+1)/2]} \Bigg] + \frac{(-1)^I}{\Gamma(\sigma I/2)} \left(\frac{u}{\pi}\right)^{(\sigma I - d')/2} [A^{\sigma I - d'}(u) - 1] \Bigg\} \\ &= (2\pi y)^{\sigma(I-1)} \tilde{D}^{(I)}_{I\sigma, d', \sigma} \end{aligned} \tag{A.10}$$

The newly formed constant $\tilde{D}^{(I)}_{\sigma I, d', \sigma}$ is finite for all $\sigma > 0$ and $d' < \sigma$, in contrast with $D^{(I)}_{d,d',\sigma}$, which is singular at $d \to \sigma I$. Subtracting next the left-hand side of the identity (A.8) multiplied by the coefficient (A.9) and combining with the previous results, we finally obtain Eq. (2.23) for all $d/\sigma = I \geqslant 2$.

Our motivation for adding and subtracting the identity (A.8) is the following. As is know, the Watson-type integral (1.7) at the borderline dimensionality $d=\sigma I$ has a leading singularity of the form (see, e.g., refs. 1 and 13) $d_*(d, \sigma)\, \tilde{\phi}^{(I-1)} \ln \tilde{\phi}$. On the other hand, when $y \to 0$ the function $F_{d,d',\sigma}(y)$ must contain the same singularity as the bulk term but with the opposite sign, in order to cancel it out and to ensure thus the analyticity of $W^{(L)}_{d,d',\sigma}(y)$ (in the case of finite L and $d' < \sigma$ there is no phase transition).

In the case $d/\sigma = 1$, which corresponds to the lower critical dimension, the finite sum in the right-hand side of Eq. (2.23) consists of just one term (with $k = 1$) and by using again the identity (A.8) with the coefficient (A.9) at $I = 1$, we obtain

$$\begin{aligned} &\lim_{d \to \sigma^+} \left\{ C_{d,d',\sigma} \left[(2\pi)^\sigma \Gamma\left(\frac{\sigma}{2}\right) \right]^{-1} - D^{(I)}_{d,\sigma} (2\pi y)^{d-\sigma} \right\} \\ &\quad = 2 \left[\sigma(4\pi)^{\sigma/2} \Gamma\left(\frac{\sigma}{2}\right) \right]^{-1} \ln(2\pi y)^\sigma \\ &\quad\quad - (4\pi)^{-\sigma/2} \int_0^\infty \frac{du}{u} \Bigg\{ E_{\sigma/2,\sigma/2}(-u^{\sigma/2}) \\ &\quad\quad - \frac{1}{\Gamma(\sigma/2)} \left(\frac{u}{\pi}\right)^{(\sigma - d')/2} [A^{\sigma - d'}(u) - 1] \Bigg\} \end{aligned} \tag{A.11}$$

The integral in the right-hand side of Eq. (A.11) is well defined and we may set it to a new constant $\tilde{D}^{(1)}_{d,d',\sigma}$; see Eq. (2.26).

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